On Approximate Solutions for Robust Convex Optimization Problems (Nonlinear Analysis and Convex Analysis)

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Jae Hyoung Lee

Department of Applied Mathematics, Pukyong National University

Gue Myung Lee

Department of Applied Mathematics, Pukyong National University

Abstract

We review our results for approximate solutions for a robust convex optimization problem with a geometric constraint, which is the face of data uncertainty. In this review, we notice that using robust optimization approach(worst-case approach), we can get an optimality theorem and duality theorems for approximate solutions for the robust convex optimization problem, and that we can extend the optimality and duality results for the convex optimization problem to a fractional optimization problem with uncertainty data.

Key words. robust convex optimization problem, robust fractional programming, approximate-solution, robust optimization approach, approximate-optimality conditions, approximate-duality theorems.

AMS subject classification 90C25 · 90C32 · 90C46.

1 Introduction

Robust convex optimization problems are to optimize convex optimization problems with data uncertainty (incomplete data) by using the worst-case approach. Here, uncertainty means that input parameter of these problems are not known exactly at the time when solution has to be determined [3].

The study of convex programs that are affected by data uncertainty ([1, 2, 3, 4, 5, 9, 10, 12]) is becoming increasingly important in optimization. Recently, the duality theory for convex programs under uncertainty via robust approach(worst-case approach) have been studied ([1, 10, 11, 12]). It was shown that primal worst equals dual best ([1, 10, 11]).

A standard form of convex optimization problem ([6, 15]) with a geometric constraint set is as follows:

$$\begin{array}{lll}(\mathrm{CP}) & \min & f(x) \\ & \mathrm{s.t.} & g_i(x) \leqq 0,\ i=1,\cdots,m, \\ & & x \in C,\end{array}$$

where $f,\ g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, m$, are convex functions and C is a closed convex cone of $\mathbb{R}^n$.

The convex optimization problem (CP) in the face of data uncertainty in the constraints can be captured by the problem

$$\begin{array}{lll}(\text{UCP}) & \min & f(x) \\ & \text{s.t.} & g_i(x,v_i) \leqq 0,\ i=1,\cdots,m, \\ & & x \in C,\end{array}$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R},\ g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{V}_i \subset \mathbb{R}^q,\ i=1,\cdots,m$.

We study an approximate optimality theorem and approximate duality theorem for the uncertain convex optimization problem (UCP) by examining its robust (worst-case) counterpart ([3])

$$\begin{array}{lll}(\text{RUCP}) & \min & f(x) \\ & \text{s.t.} & g_i(x,v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i=1,\cdots,m, \\ & & x \in C.\end{array}$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R},\ g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is the uncertain parameter which belongs to the set $\mathcal{V}_i \subset \mathbb{R}^q,\ i=1,\cdots,m$. Clearly, $A := \{x \in C \mid g_i(x,v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i=1,\ldots,m\}$ is the feasible set of (RUCP).

Let $\epsilon \geqq 0$. Then $\bar{x}$ is called an approximate solution of (RUCP) if for any $x \in A$,

$$f(x) \geqq f(\bar{x}) - \epsilon.$$

Recently, Jeyakumar and Li [10] has showed that when $C = \mathbb{R}^n$ and $\epsilon = 0$, Lagrangian strong duality holds between s robust counterpart and an optimistic counterpart for robust convex optimization problem in the face of data uncertainty via robust optimization under a new robust characteristic cone constraint qualification (RCCCQ) that

$$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \text{epi}\Big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\Big)^*$$

is convex and closed.

In this paper, we give approximate optimality theorem for (RUCP) under the following constraint qualification:

$$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \text{epi}\Big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\Big)^* + C^* \times \mathbb{R}_+$$

is convex and closed. For approximate solutions of (RUCP), we formulate a Wolfe type dual problem for the primal one and give approximate weak duality and approximate strong duality between the primal problem and its Wolfe type dual problem, which hold under a weakened constraint qualification. Moreover, we notice that we can extend the optimality and duality results for (RUCP) to a fractional optimization problem with uncertainty data.

2 Definitions and Notations

Let us first recall some definitions and notations which will be used throughout this paper. $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geqq 0\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We say the set A is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$. Let

f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$. Here, f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by dom$f$, that is, dom$f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$. The epigraph of f, epi$f$, is defined as epi$f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqq r\}$, and f is said to be convex if for all $\mu \in [0, 1]$,

$$f((1-\mu)x + \mu y) \leqq (1-\mu)f(x) + \mu f(y)$$

for all $x, y \in \mathbb{R}^n$, equivalently epi$f$ is convex. The function f is said to be concave whenever $-f$ is convex. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle \leq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}, \text{ if } x \in \mathrm{dom} f, \\ \emptyset, \text{ otherwise.} \end{cases}$$

More generally, for any $\epsilon \geq 0$, the ϵ-subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle \leq f(y) - f(x) + \epsilon,\ \forall y \in \mathbb{R}^n\}, \text{ if } x \in \mathrm{dom} f, \\ \emptyset, \text{ otherwise.} \end{cases}$$

As usual, for any proper convex function g on $\mathbb{R}^n$, its conjugate function $g^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by for any $x^* \in \mathbb{R}^n$, $g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) \mid x \in \mathbb{R}^n\}$. For details of conjugate function, see [15]. Given a set $A \subset \mathbb{R}^n$, we denote the closure, the convex hull, and the conical hull generated by A, by cl$A$, co$A$, and cone$A$, respectively. The normal cone $N_C(x)$ to C at x is defined by

$$N_C(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x\rangle \ \leqq\ 0, \text{ for all } y \in C\},$$

and let $\epsilon \geqq 0$, then the ϵ-normal set $N_C^\epsilon(x)$ to C at x is defined by

$$N_C^\epsilon(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x\rangle \ \leqq\ \epsilon, \text{ for all } y \in C\}.$$

When C is a closed convex cone in $\mathbb{R}^n$, we denote $N_C(0)$ by C^* and call it the negative dual cone of C.

3 Approximate Optimality Theorem

Slightly extending Theorem 2.4 in [10] to a robust convex optimization problem with a geometric constraint, we can obtain the following lemma in [12], which is the robust version of Farkas Lemma for convex functions in [8]:

Lemma 3.1. [12] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is a convex function. Let C be a closed convex cone of $\mathbb{R}^n$. Let $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \ldots, m$, and let $A := \{x \in C \mid g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} \neq \emptyset$. Then the following statements are equivalent:*

(i) $\{x \in C \mid g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} \subseteq \{x \in \mathbb{R}^n \mid f(x) \geqq 0\}$;

(ii) $(0, 0) \in \mathrm{epi} f^* + \mathrm{cl\,co}\big(\bigcup_{v_i \in \mathcal{V}_i,\ \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+\big)$.

Using Lemma 3.1, we can obtain the following theorem in [12]:

Theorem 3.1. [12] *Let $\bar{x} \in A$ and let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \dots, m$, be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is convex for each fixed $v_i \in \mathcal{V}_i$. Suppose that $\bigcup_{v_i \in \mathcal{V}_i,\ \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$ is closed and convex. Then $\bar{x}$ is an approximate solution of* (RUCP) *if and only if there exist $\bar{\lambda}_i \geqq 0$ and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$, such that for any $x \in C$,*

$$f(x) + \sum_{i=1}^m \bar{\lambda}_i g_i(x, \bar{v}_i) \geqq f(\bar{x}) - \epsilon.$$

Using Lemma 3.1, we can obtain the following approximate optimality theorem for approximate solution of (RUCP) which is in [12].

Theorem 3.2. [12] **(Approximate Optimality theorem)** *Let $\bar{x} \in A$ and let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \dots, m$, be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is convex for each fixed $v_i \in \mathcal{V}_i$. Suppose that $\bigcup_{v_i \in \mathcal{V}_i,\ \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$ is closed and convex. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an approximate solution of* (RUCP)*;*

(ii) $(0, \epsilon - f(\bar{x})) \in \mathrm{epi} f^* + \displaystyle\bigcup_{v_i \in \mathcal{V}_i,\ \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$;

(iii) *There exist $\bar{v}_i \in \mathcal{V}_i$, $\bar{\lambda}_i \geqq 0$, $i = 1, \dots, m$, and $\epsilon_i \geqq 0$, $i = 0, 1, \dots, m+1$ such that*

$$0 \in \partial_{\epsilon_0} f(\bar{x}) + \sum_{i=1}^m \partial_{\epsilon_i}(\bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)) + N_C^{\epsilon_{m+1}}(\bar{x})$$

$$\text{and} \quad \sum_{i=0}^{m+1} \epsilon_i - \epsilon = \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)$$

As usual convex program, the dual problem of (RUCP) is sometimes more treatable than (RUCP). So, we formulate a dual problem (RLD) for (RUCP) as follows([12]):

$$\begin{array}{lll} \text{(RLD)} & \text{Maximize}_{(x,v,\lambda)} & f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i) \\ & \text{subject to} & 0 \in \partial_{\epsilon_0} f(x) + \sum_{i=1}^m \partial_{\epsilon_i} \lambda_i g_i(x, v_i) + N_C^{\epsilon_{m+1}}(x), \\ & & \lambda_i \geqq 0,\ v_i \in \mathcal{V}_i,\ i = 1, \dots, m, \\ & & \sum_{i=0}^{m+1} \epsilon_i \leqq \epsilon. \end{array}$$

When $\epsilon = 0$, and $g_i(x, v_i) = g_i(x)$, $i = 1, \dots, m$, (RUCP) becomes (CP), and (RLD) collapses to the Wolfe dual problem (D) for (CP) as follows:

$$\begin{array}{lll} \text{(D)} & \text{Maximize}_{(x,\lambda)} & f(x) + \sum_{i=1}^m \lambda_i g_i(x) \\ & \text{subject to} & \nabla f(x) + \sum_{i=1}^m \nabla \lambda_i g_i(x) + N_C(x) = 0, \\ & & \lambda_i \geqq 0,\ i = 1, \dots, m. \end{array}$$

Now, we prove approximate weak and approximate strong duality theorems which hold between (RUCP) and (RLD) which are in [12].

Theorem 3.3. [12] **(Approximate Weak Duality Theorem)** *For any feasible x of* (RUCP) *and any feasible (y, v, λ) of* (RLD),

$$f(x) \geqq f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \epsilon.$$

Theorem 3.4. [12] **(Approximate Strong Duality Theorem)** *Let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \dots, m$, be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is a convex for each fixed $v_i \in \mathcal{V}_i$. Suppose that*

$$\bigcup_{v_i \in \mathcal{V}_i,\ \lambda_i \geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$$

is closed. If $\bar{x}$ is an approximate solution of (RUCP), *then there exist $\bar{\lambda} \in \mathbb{R}^m_+$ and $\bar{v} \in \mathbb{R}^q$ such that $(\bar{x}, \bar{v}, \bar{\lambda})$ is a 2ϵ-solution of* (RLD).

4 Robust Fractional Optimization Problem

In this chapter, we notice that we can extend the optimality and duality results for the convex optimization problem to a fractional optimization problem with uncertainty data.

Consider the following standard form of fractional optimization problem with a geometric constraint set:

$$\begin{array}{lll} \text{(FP)} & \min & \dfrac{f(x)}{g(x)} \\ & \text{s.t.} & h_i(x) \leqq 0,\ i = 1, \cdots, m, \\ & & x \in C, \end{array}$$

where $f, h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, m$, are convex functions, C is a closed convex cone of $\mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ is a concave function such that for any $x \in C$, $f(x) \geqq 0$ and $g(x) > 0$.

The fractional optimization problem (FP) in the face of data uncertainty in the constraints can be captured by the problem:

$$\begin{array}{lll} \text{(UFP)} & \min & \displaystyle\max_{(u,v) \in \mathcal{U} \times \mathcal{V}} \frac{f(x, u)}{g(x, v)} \\ & \text{s.t.} & h_i(x, w_i) \leqq 0,\ i = 1, \cdots, m, \\ & & x \in C, \end{array}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $f(\cdot, u)$ and $h_i(\cdot, w_i)$ are convex, and $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $g(\cdot, v)$ is concave, and $u \in \mathbb{R}^p$, $v \in \mathbb{R}^p$ and $w_i \in \mathbb{R}^q$ are uncertain parameters which belongs to the convex and compact uncertainty sets $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$ and $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1, \cdots, m$, respectively.

We study approximate optimality theorems and approximate duality theorems for the uncertain fractional optimization problem (UFP) by examining its robust (worst-case) counterpart ([3]):

$$\begin{array}{lll} \text{(RFP)} & \min & \displaystyle\max_{(u,v) \in \mathcal{U} \times \mathcal{V}} \frac{f(x, u)}{g(x, v)} \\ & \text{s.t.} & h_i(x, w_i) \leqq 0,\ \forall w_i \in \mathcal{W}_i,\ i = 1, \cdots, m, \\ & & x \in C. \end{array}$$

Clearly, $A := \{x \in C \mid h_i(x, w_i) \leqq 0,\ \forall w_i \in \mathcal{W}_i,\ i = 1, \ldots, m\}$ is the feasible set of (RFP).

Let $\epsilon \geqq 0$. Then $\bar{x}$ is called an approximate solution of (RFP) if for any $x \in A$,

$$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon.$$

Using parametric approach, we transform the problem (RFP) into the robust non-fractional convex optimization problem $(\mathrm{RNCP})_r$ with a parameter $r \in \mathbb{R}_+$:

$$\begin{array}{lll} (\mathrm{RNCP})_r & \min & \max_{u\in\mathcal{U}} f(x,u) - r \min_{v\in\mathcal{V}} g(x,v) \\ & \text{s.t.} & h_i(x,w_i) \leqq 0,\ \forall w_i \in \mathcal{W}_i,\ i = 1, \cdots, m, \\ & & x \in C. \end{array}$$

Let $\epsilon \geqq 0$. Then $\bar{x}$ is called an approximate solution of $(\mathrm{RNCP})_r$ if for any $x \in A$,

$$\max_{u\in\mathcal{U}} f(x,u) - r \min_{v\in\mathcal{V}} g(x,v) \geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - r \min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon.$$

Now we give the following relation between approximate solution of (RFP) and $(\mathrm{RNCP})_{\bar{r}}$, which is in [13].

Lemma 4.1. [13] *Let $\bar{x} \in A$ and let $\epsilon \geqq 0$. If $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon \geqq 0$, then the following statements are equivalent:*

(i) *$\bar{x}$ is an approximate solution of* (RFP)*;*

(ii) *$\bar{x}$ is an $\bar{\epsilon}$-solution of* $(\mathrm{RNCP})_{\bar{r}}$*, where $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ and $\bar{\epsilon} = \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v)$.*

From Lemma 4.1, we can get the following theorem in [13] with a similar way to Theorem 3.1.

Theorem 4.1. [13] *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ be functions such that for any $u \in \mathcal{U}$, $f(\cdot,u)$ and for each $w_i \in \mathcal{W}_i$, $h_i(\cdot,w_i)$, $i = 1, \ldots, m$, are convex functions, and for any $x \in \mathbb{R}^n$, $f(x,\cdot)$ is concave function. Let $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be a function such that for any $v \in \mathcal{V}$, $g(\cdot,v)$ is a concave function, and for all $x \in \mathbb{R}^n$, $g(x,\cdot)$ is a convex function. Let $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$ and $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$. Let $\bar{x} \in A$ and let $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$. Suppose that $\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot,w_i))^* + C^* \times \mathbb{R}_+$ is closed and convex. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an approximate solution of* (RFP)*;*

(ii) *there exist $\bar{u} \in \mathcal{U}$, $\bar{v} \in \mathcal{V}$, $\bar{w}_i \in \mathcal{W}_i$ and $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, such that for any $x \in C$,*

$$f(x,\bar{u}) - \bar{r} g(x,\bar{v}) + \sum_{i=1}^{m} \bar{\lambda}_i h_i(x,\bar{w}_i) \geqq 0.$$

Using Lemma 4.1, we can establish approximate optimality theorems ([13]) for approximate solutions for the robust fractional optimization problem with a similar way to Theorem 3.2.

Theorem 4.2. [13] **(Approximate Optimality theorem)** *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ be functions such that for any $u \in \mathcal{U}$, $f(\cdot,u)$ and for each $w_i \in \mathcal{W}_i$, $h_i(\cdot,w_i)$, $i = 1, \ldots, m$, are convex functions. Let $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be a function such that for any $v \in \mathcal{V}$, $g(\cdot,v)$ is a concave function. Let $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$ and $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$. Let $\bar{x} \in A$ and let $\epsilon \geqq 0$. Let $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$.*

If $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} < \epsilon$*, then* $\bar{x}$ *is an approximate solution of* (RFP). *If* $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} \geqq \epsilon$ *and* $\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot,w_i))^* + C^* \times \mathbb{R}_+$ *is closed and convex, then the following statements are equivalent:*

(i) $\bar{x}$ *is an approximate solution of* (RFP)*;*

(ii) *There exist* $\bar{w}_i \in \mathcal{W}_i$ *and* $\bar{\lambda}_i \geqq 0$, $i = 1,\ldots,m$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$ *and* $\epsilon_i \geqq 0$, $i = 1,\ldots,m+1$ *such that*

$$
\begin{aligned}
&0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(\bar{x}) + \partial_{\epsilon_0^2}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x}) \\
&\quad + N_C^{\epsilon_{m+1}}(\bar{x}), \\
&\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) = \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) \quad \text{and} \\
&\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) = \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x},\bar{w}_i).
\end{aligned}
$$

If for all $x \in \mathbb{R}^n$, $f(x,\cdot)$ is concave, and for all $x \in \mathbb{R}$, $g(x,\cdot)$ is convex, then using Lemma 4.1, we can obtain the following characterization of approximate solution for (RFP) which is in [13].

Theorem 4.3. [13] **(Approximate Optimality theorem)** *Let* $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ *and* $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1,\ldots,m$, *be functions such that for any* $u \in \mathcal{U}$, $f(\cdot,u)$ *and for each* $w_i \in \mathcal{W}_i$, $h_i(\cdot,w_i)$ *are convex functions, and for all* $x \in \mathbb{R}^n$, $f(x,\cdot)$ *is concave function. Let* $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ *be a function such that for any* $v \in \mathcal{V}$, $g(\cdot,v)$ *is concave, and for all* $x \in \mathbb{R}$, $g(x,\cdot)$ *is convex. Let* $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$ *and* $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1,\ldots,m$. *Let* $\bar{x} \in A$ *and let* $\epsilon \geqq 0$. *Let* $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$. *If* $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} < \epsilon$*, then* $\bar{x}$ *is an approximate solution of* (RFP). *If* $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} \geqq \epsilon$ *and* $\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot,w_i))^* + C^* \times \mathbb{R}_+$ *is closed and convex, then the following statements are equivalent:*

(i) $\bar{x}$ *is an approximate solution of* (RFP)*;*

(ii) *There exist* $\bar{u} \in \mathcal{U}$, $\bar{v} \in \mathcal{V}$, $\bar{w}_i \in \mathcal{W}_i$, $\bar{\lambda}_i \geqq 0$, $i = 1,\ldots,m$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$ *and* $\epsilon_i \geqq 0$, $i = 1,\ldots,m+1$, *such that*

$$
\begin{aligned}
&0 \in \partial_{\epsilon_0^1}(f(\cdot,\bar{u}))(\bar{x}) + \partial_{\epsilon_0^2}(-\bar{r}g(\cdot,\bar{v}))(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x}) \\
&\quad + N_C^{\epsilon_{m+1}}(\bar{x}), \\
&\max_{u\in\mathcal{U}} f(\bar{x},u) - \min_{v\in\mathcal{V}} \bar{r} g(\bar{x},v) = \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) \text{ and} \\
&\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) \leqq \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x},\bar{w}_i).
\end{aligned}
$$

Following the approach in [7], we formulate a dual problem (RFD) for (RFP) as follows ([13]) :

$$
\begin{aligned}
\text{(RFD)}\quad & \max \quad r \\
& \text{s.t.} \quad 0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(x) + \partial_{\epsilon_0^2}(-r\min_{v\in\mathcal{V}} g(\cdot,v))(x) \\
& \qquad + \sum_{i=1}^{m} \partial_{\epsilon_i}(\lambda_i h_i(\cdot,w_i))(x) + N_C^{\epsilon_{m+1}}(x), \\
& \qquad \max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) \geqq \epsilon \min_{v\in\mathcal{V}} g(x,v), \\
& \qquad \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i - \epsilon\min_{v\in\mathcal{V}} g(x,v) \leqq \sum_{i=1}^{m}\lambda_i h_i(x,w_i), \\
& \qquad r \geqq 0,\ w_i \in \mathcal{W}_i,\ \lambda_i \geqq 0,\ i=1,\ldots,m, \\
& \qquad \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_i \geqq 0,\ i=1,\ldots,m+1.
\end{aligned}
$$

Clearly, $F := \{(x,w,\lambda,r) \mid 0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(x) + \partial_{\epsilon_0^2}(-r\min_{v\in\mathcal{V}} g(\cdot,v))(x) + \sum_{i=1}^{m}\partial_{\epsilon_i}(\lambda_i h_i(\cdot,w_i))\ (x) + N_{\mathbb{R}_+}^{\epsilon_2}(x),\ \max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) \geqq \epsilon g(x,v),\ \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i - \epsilon\min_{v\in\mathcal{V}} g(x,v) \leqq \sum_{i=1}^{m}\lambda_i h_i(x,w_i),\ r \geqq 0,\ w_i \in \mathcal{W}_i,\ \lambda_i \geqq 0,\ \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_i \geqq 0,\ i=1,\ldots,m,\ \epsilon_{m+1} \geqq 0\}$ is the feasible set of (RFD).

Let $\epsilon \geqq 0$. Then $(\bar{x},\bar{w},\bar{\lambda},\bar{r})$ is called an approximate solution of (RFD) if for any $(y,w,\lambda,r) \in F$, $\bar{r} \geqq r - \epsilon$.

When $\epsilon = 0$, $\max_{u\in\mathcal{U}} f(x,u) = f(x)$, $\min_{v\in\mathcal{V}} g(x,v) = g(x)$ and $h_i(x,w_i) = h_i(x)$, $i = 1,\ldots,m$, (RFP) becomes (FP), and (RFD) collapses to the Mond-wier type dual problem (FD) for (FP) as follows ([14]):

$$
\begin{aligned}
\text{(FD)}\quad & \max \quad r \\
& \text{s.t.} \quad 0 \in \partial f(x) + \partial(-rg)(x) + \sum_{i=1}^{m}\partial\lambda_i h_i(x) + N_C(x), \\
& \qquad f(x) - rg(x) \geqq 0,\ \lambda_i h_i(x) \geqq 0, \\
& \qquad r \geqq 0,\ \lambda_i \geqq 0,\ i=1,\ldots,m.
\end{aligned}
$$

Now, we prove approximate weak and approximate strong duality theorems which hold between (RFP) and (RFD).

Theorem 4.4. [13] **(Approximate Weak Duality Theorem)** *For any feasible x of* (RFP) *and any feasible (y,w,λ,r) of* (RFD),

$$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq r - \epsilon.$$

Theorem 4.5. [13] **(Approximate Strong Duality Theorem)** *Suppose that*

$$\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \operatorname{epi}(\sum_{i=1}^{m}\lambda_i g_i(\cdot,w_i))^* + C^* \times \mathbb{R}_+$$

is closed. If $\bar{x}$ is an approximate solution of (RFP) *and* $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon \geqq 0$, *then there exist $\bar{w} \in \mathbb{R}^q$, $\bar{\lambda} \in \mathbb{R}_+^m$ and $\bar{r} \in \mathbb{R}_+$ such that $(\bar{x},\bar{w},\bar{\lambda},\bar{r})$ is a 2ϵ-solution of* (RFD).

Remark 4.1. *Using the optimality conditions of Theorem 4.2, robust fractional dual problem* (RFD) *for a robust fractional problem* (RFP) *in the convex constraint functions with uncertainty is formulated. However, when we formulated the dual problem using optimality condition in Theorem 4.3, we could not know whether approximate weak duality theorem is established, or not. It is our open question.*

References

[1] A. Beck and A. Ben-Tal, Duality in robust optimization: primal worst equals dual best, *Oper. Res. Lett.,* **37** (2009), 1-6.

[2] A. Ben-Tal, L. E. Ghaoui and A. Nemirovski, *Robust Optimzation*, Princeton Series in Applied Mathematics, Priceton University Press, Priceton, NJ (2009).

[3] A. Ben-Tal and A. Nemirovski, A selected topics in robust convex optimization, *Math. Program.,* Ser B, **112** (2008), 125-158.

[4] A. Ben-Tal and A. Nemirovski, Robust optimization-methodology and applications, *Math. Program.,* Ser B, **92** (2002), 453-480.

[5] A. Ben-Tal and A. Nemirovski, Robust solutions to uncertain linear programs, *Oper. Res. Lett.*, **25** (1999), 1-13.

[6] S. Boyd and L. Vandenberghe, *Convex optimization*, Cambridge University Press, Cambridge, 2004.

[7] P. Gupta, A. Mehra, S. Shiraishi and K. Yokoyama, ϵ-Optimality for minimax programming problems. *J. Nonlinear Convex Anal.*, **7**(2) (2006), 277-288.

[8] V. Jeyakumar, G. M. Lee and N. Dinh, New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs, *SIAM J. Optim.*, **14**(2) (2003), 534-547.

[9] V. Jeyakumar and G. Y. Li, Robust duality for fractional programming problems with constraint-wise data uncertainty, *J. Optim. Theory Appl.*, **151**(2) (2011), 292-303.

[10] V. Jeyakumar an G. Y. Li, Strong duality in robust convex programming: complete characterizations, *SIAM J. Optim.,* **20** (2010), 3384-3407.

[11] V. Jeyakumar, G. Y. Li and G. M. Lee, Robust duality for generalized convex programming problems under data uncertainty, *Nonlinear Anal.*, **75**(3) (2012), 1362-1373.

[12] J. H. Lee and G. M. Lee, On ϵ-solutions for convex optimization problems with uncertainty data, *Positivity* **16**(3) (2012), 509-526.

[13] J. H. Lee and G. M. Lee, On ϵ-solutions for robust fractional optimization problems, *J. Inequal. Appl.* **2014**:501 (2014), doi:10.1186/1029-242X-2014-501.

[14] J. C. Liu, Y. Kimura and K. Tanaka, Three Types Dual Model for Minimax Fractional Programming. *Comput. Math. Appl.*, **38**(7-8) (1999), 143-155.

[15] R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, N. J, 1970.

(Jae Hyoung Lee) Department of Applied Mathematics
Pukyong National University
Busan 48513
Korea
E-mail address: mc7558@naver.com

(Gue Myung Lee) Department of Applied Mathematics
Pukyong National University
Busan 48513
Korea
E-mail address: gmlee@pknu.ac.kr